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Metastable currents in loops with Josephson junctions

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Abstract
The problem of persistent currents in loops interrupted by Josephson junctions is considered. We prove that persistent currents are related to local minimizers of the Ginzburg Landau energy functional. Although the order parameter is discontinuous at the junction, we show that the local minimizers are related to the homotopy types of the domain. Therefore, persistent currents will occur in any multiply connected domain if the junction strength is weak enough.

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1. Introduction and formulation

The phenomenon of persistent currents is one of the most striking aspects of superconductivity. It is often used, together with zero electrical resistivity and the expulsion of magnetic fields (Meissner effect), to demonstrate the exceptional nature of superconductivity. Persistent currents are currents that flow without any external driving force in multiply connected superconducting samples such as rings or solid tori. These currents persist without appreciable dissipation for extremely long periods of time. They were discovered by K Onnes immediately after he observed the phase transition into the superconducting state. The relation of persistent currents to magnetic flux quantization was established in a classical experiment by Deaver and Fairbanks [2]. Since there is no driving force, persistent currents are associated with metastable states.

Inspired by the Deaver–Fairbanks experiment, Goldman *et al* [3] studied metastable currents in superconducting rings interrupted by a thin normal layer (Josephson junction). Since the phase change across the junction can be large, and since it is the circulation of the phase gradient in the loop that is related to the magnetic flux, it is not clear if the persistent current effect will also occur in the interrupted junction (for technical reasons Goldman *et al* used a ring with two junctions, but this does not alter the general picture). Later, we shall argue from a more mathematical point of view why the existence of metastable currents in rings with junctions is not obvious at all.

The connection between persistent currents and flux quantization hints that the mathematical structure behind this effect is related to the homotopy characteristics of the domain. In the absence of a driving force the global minimizer of the free energy is of course the constant state, where no current flows. Therefore, persistent currents must be local minimizers. We use the Ginzburg Landau (GL) model for the energy:

$$G(u, A) = \int_\Omega \left(|(\mathrm{i}\nabla - A)u|^2 + \frac{\kappa^2}{2}(|u|^2 - 1)^2 \right) \mathrm{d}x + \int_{R^n} |\nabla \times A|^2 \, \mathrm{d}x. \tag{1}$$

Here u is the complex-valued order parameter, A is the magnetic vector potential, κ is the GL parameter and Ω is the domain occupied by the superconducting sample. The derivation of the nondimensional model (1) can be found in numerous texts such as [1, 7], etc. The functional G is to be minimized in an appropriate function space that will be defined below. The difficulty is that the functions in the relevant space are not necessarily smooth. Since homotopy is defined for continuous functions, there is a conflict between the physical intuition (local minimizers should exist, and they are associated with homotopy types) and the mathematical setup of the optimization problem. This conflict was resolved in [6] where the authors used a result of White [8] that enables us to endow even certain nonsmooth functions with the homotopy type. Therefore, the case of persistent currents in 'clean' rings is now understood.

The goal of the current paper is to derive a similar theory also for the case of persistent currents in rings with junctions. We must first decide on a model for the junctions. Recall that the key idea behind the Josephson effect is that the current flowing through the junction is a periodic function of the phase jump across it. The standard approximation is

$$J \propto \sin[\phi], \tag{2}$$

where J is the supercurrent, ϕ is the phase of the complex-valued u and $[\cdot]$ denotes a jump of a quantity across the junction. In order to work effectively with the Josephson condition, we include it in the GL energy function. A useful way to do it is to model the junction by a weak link. There are a number of ways of doing that. One is to assume that there is a constriction, namely, the thickness of the ring (or torus) becomes very small right at (and near) the junction. Another way is to assume that the sample is 'dirty' near the junction, i.e. the coherence length is very small there. Mathematically, the two models are similar. The problem of a GL model with a weak link was studied rigorously in [5]. It was shown there that under certain canonical scaling, the energy of the ring (torus) plus junction becomes

$$G_J(u, A) = \int_{\tilde\Omega} \left(|(\mathrm{i}\nabla - A)u|^2 + \frac{\kappa^2}{2}(|u|^2 - 1)^2 \right) \mathrm{d}x + \int_{R^n} |\nabla \times A|^2 \, \mathrm{d}x + b\int_\Gamma [u]^2. \tag{3}$$

Here we denote the domain minus the junction by $\tilde\Omega$, and the junction surface is denoted by Γ. See figure 1. The parameter b is the inverse of the junction strength; thus a large b models a weak link. The functional G_J is minimized in a function space defined over $\tilde\Omega$, i.e. the functions need not be continuous across the junction Γ.

Our problem, therefore, is to show that the energy G_J has local minimizers that can be classified by their homotopy type. The difficulty is now more severe than that in the first (clean ring) model (1). In the clean model, we must work in function spaces that include nonsmooth functions. However, we can expect the minimizer to be at least continuous (since stationary functions of functionals like (1) are typically smooth). On the other hand, even in the most optimistic scenario, the local minimizers of (3) are surely going to have discontinuity at the junction Γ. Nevertheless, we shall show that for sufficiently large b, i.e. for sufficiently weak junctions, there do exist local minimizers that are classified by their homotopy type.

In the following section, we shall present an explicit calculation in a simple one-dimensional model. The main result of this paper will be stated and proved in section 3.

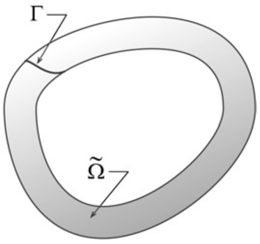

Figure 1. A sketch of the ring geometry with the junction Γ.

This result will be discussed and extended in several ways in section 4. We point out that the functionals G and G_J above and the functionals e^b, E^b, E^∞ and G_J^b below are all invariant under certain gauge transformations. Therefore, the solutions derived here are unique only up to such a transformation.

2. A one-dimensional toy problem

The purpose of this section is to present a very simple one-dimensional example of a solution that exhibits persistent (metastable) currents. The computations will be explicit; in fact the example is not new, and the essential details were already clear to Goldman *et al* [3] and later to Ouboter [4]. Nevertheless, we choose to start with this simple toy problem since it reveals very clearly the nature of the solution and the connection with the homotopy type. This will enhance the understanding of the following section.

Consider a loop of length 2π, parameterized by its arclength s. Since there is no applied magnetic field, we select a gauge where $A = 0$. Also, for the sake of simplicity we assume that the order parameter $u(s)$ takes values in the unit circle S^1, i.e. $u : S^1 \to S^1$. The junction is located at $s = 0$, and u can have a jump discontinuity there. The cost function e^b is defined by

$$e^b(u) = \int_0^{2\pi} |u'|^2 \, \mathrm{d}s + b([u])^2. \tag{4}$$

As usual we use $[\cdot]$ to denote the jump of a quantity across the junction at $s = 0$, b is a positive parameter and $u' = \mathrm{d}u/\mathrm{d}s$. We shall show through an explicit construction that for every integer m there exists a number b_m, such that there exists exactly one stable nontrivial minimizer u_n for all integers n, $0 < n \leqslant m$.

Since u takes values in S^1, it can be expressed as $u(s) = \exp(\mathrm{i}\phi(s))$. The functional e^b is invariant under the transformation $u \to u\exp(\mathrm{i}\alpha)$ for any constant α. It is convenient to fix the phase ϕ at the origin, say by $\phi(0) = 0$. Inserting the polar representation of u into equation (4) gives

$$F^b(\phi) = \int_0^{2\pi} (\phi')^2 \, \mathrm{d}s + b(2 - 2\cos(\phi(2\pi))). \tag{5}$$

The Euler Lagrange equation of F^b is simply $\phi'' = 0$, implying that the current $J \equiv \phi'(s)$ is constant in the loop. Equating the first variation of F^b to zero also provides a jump condition across the junction at $s = 0$:

$$\phi'(2\pi) = \phi'(0) = J = b\sin(\phi(2\pi)). \tag{6}$$

Integrating the equation $\phi' = J$ along the loop gives $\phi(2\pi) = 2\pi J$. Inserting this result into equation (6), we obtain

$$J = b\sin(2\pi J). \tag{7}$$

The optimization problem was thus reduced to the algebraic problem (7). Each solution of this equation is associated with a critical point of the functional F^b. Before considering the solutions, we derive a criterion for their stability, that is, a criterion for a solution to be a minimizer. For this purpose, we compute the second variation of F^b:

$$\delta^2 F^b(\phi, \beta) = \int_0^{2\pi} (\beta')^2 \, \mathrm{d}s + b\cos(\phi(2\pi))\beta^2. \tag{8}$$

Therefore a solution ϕ is a minimizer when $\cos(\phi(2\pi)) > 0$, while the solution is not a minimizer if $\cos(\phi(2\pi)) < 0$.

We are now in a position to examine the set of minimizers. It is clear from trigonometric considerations that in each period of the sine function there are at most two solutions to equation (7): the first of them is stable and the second is unstable. The number of such solutions is an increasing function of b.

3. Homotopy classification of metastable currents

We shall now show that the simple situation explained in the previous section can be generalized to arbitrary geometries in R^2 and R^3 with a nontrivial homotopy structure. Our results are based on [6] where the idea of classification of minimizers by the homotopy type was introduced. Therefore, to simplify the presentation, we shall analyse in some detail a special case of a more general problem, and highlight the new features of the problem under consideration here. Later, we shall comment on a number of extensions of the special case.

Let Ω be a domain in R^n, $n = 2$ or $n = 3$, that is topologically equivalent to the solid torus. Let Γ be an $(n-1)$-dimensional surface that cuts the torus so that the domain $\tilde{\Omega} = \Omega \backslash \Gamma$ is simply connected. We assume that both Ω and Γ are smooth. We further define the following functional:

$$E^b(u) = \int_{\tilde{\Omega}} (|\nabla u|^2 + b(|u|^2 - 1)^2) \, \mathrm{d}x + b \int_\Gamma [u]^2 \, \mathrm{d}x. \tag{9}$$

Here, as elsewhere in this paper, $[\cdot]$ denotes the jump of the relevant quantity across the interface Γ.

The expected jump of u across the junction Γ requires a careful definition of the relevant function spaces. We use $H^1(\tilde{\Omega})$ to denote the Sobolev space of complex-valued functions in $L_2(\tilde{\Omega})$ whose (weak) derivatives are also in $L_2(\tilde{\Omega})$. Similarly, we define the space $H^1(\Omega)$. Obviously, if $v \in H^1(\Omega)$ then also $v \in H^1(\tilde{\Omega})$ but not necessarily the other way round. It is also useful to consider the spacial spaces of functions taking values in S^1. We denote them by $H^1(\tilde{\Omega}, S^1)$ and $H^1(\Omega, S^1)$ depending on the domain of definition. We naturally seek minimizers of E^b in the space $H^1(\tilde{\Omega})$. The global minimizer is the constant function (of amplitude 1). However, we shall show that when b is large enough there exist also nontrivial local minimizers.

Since we are interested in the case of large b, it is useful to introduce an additional functional that is expected to capture the behaviour in the limit $b \to \infty$:

$$E^\infty(u) = \int_\Omega |\nabla u|^2 \, \mathrm{d}x. \tag{10}$$

The problem of finding local minimizers for E^∞ in the function space $H^1(\Omega, S^1)$ was considered in [6]. In particular, it was shown there that for each integer m there exists a local minimizer u_m of E^∞ in this space which is of the homotopy type m. The proof relies on the partition of $H^1(\Omega, S^1)$ into homotopy types (the integers in the current case) and a deep result of White [8], who showed how to define a homotopy type even for nonsmooth functions in $H^1(\Omega, S^1)$. Furthermore, the local minimizers u_m are isolated (up to the usual multiplication by $\exp(i\alpha)$ for a fixed α). Note that the functions u_m are in the space $H^1(\Omega, S^1)$, i.e. they do not have a jump discontinuity across Γ.

Our main result is the following theorem that uses the local minimizers u_m to establish the existence of local minimizers $\tilde{u}_m^b$ for E^b.

Theorem 1. *For each integer m there exists a number b_m, such that for all $b > b_m$ the functional E^b possesses a local minimizer u_m^b. Moreover, the sequence u_m^b converges strongly in $H^1(\tilde{\Omega})$ to u_m.*

Proof. For each function in $H^1(\Omega, S^1)$ of a given homotopy type m, there exists a ball of radius γ_m such that all the functions in $H^1(\Omega, S^1)$ within that ball are of the homotopy type m. Consider the problem of minimizing E^b among all functions u in $H^1(\tilde{\Omega})$ satisfying

$$\|u - u_m\|_{H^1(\tilde{\Omega})} \leqslant \gamma_m. \tag{11}$$

Since the set is closed, a minimizer exists, and we denote it by u_m^b. The following inequalities obviously hold:

$$E^b\left(u_m^b\right) \leqslant E^b(u_m) = E^\infty(u_m). \tag{12}$$

It remains to show that u_m^b is actually inside the ball defined by (11), and therefore it is a local minimizer in the metric of $H^1(\tilde{\Omega})$. Fixing m, the sequence $\left\{u_m^b\right\}$ is, thanks to inequality (12), uniformly bounded in $H^1(\tilde{\Omega})$. Therefore it weakly converges there to a limit $\tilde{u}_m$ (of course, there could be more than one weakly convergent subsequence, but this technicality can be handled as in [6]). Moreover, a lower semi-continuity argument implies that the chain (12) can be extended into

$$\tilde{E}^\infty(\tilde{u}_m) \leqslant E^b\left(u_m^b\right) \leqslant E^b(u_m) = E^\infty(u_m), \tag{13}$$

where $\tilde{E}^\infty(u) = \int_{\tilde{\Omega}} |\nabla u|^2 \,\mathrm{d}x$. Our next step is to show that $\tilde{u}_m$ is actually in the space $H^1(\Omega, S^1)$, and then $\tilde{E}^\infty(\tilde{u}_m) = E^\infty(\tilde{u}_m)$. Note first that the inequality

$$\left\|u_m^b\right\|_{H^1(\tilde{\Omega})} \leqslant C_m \tag{14}$$

implies $\int_{\tilde{\Omega}} \left(\left|u_m^b\right|^2 - 1\right)^2 \mathrm{d}x \leqslant C_m/b$. Therefore, $\left|u_m^b\right|$ converges a.e. to 1, and thus $\tilde{u}_m \in H^1(\tilde{\Omega}, S^1)$. To show that we can remove the tilde from over Ω, we multiply ∇u_m^b by a smooth test function w and integrate over $\tilde{\Omega}$. An integration by parts gives

$$\int_{\tilde{\Omega}} u_m^b \nabla w \,\mathrm{d}x = -\int_{\tilde{\Omega}} \nabla u_m^b w \,\mathrm{d}x + \int_\Gamma w\left[u_m^b\right]\nu \,\mathrm{d}x, \tag{15}$$

where ν is normal to the interface Γ. The estimate (12) implies that $\int_\Gamma w\left[u_m^b\right] \mathrm{d}x \to 0$ as $b \to \infty$. Also, the weak convergence of u_m^b to $\tilde{u}_m$ enables us to pass to the limit in (15). We thus obtain

$$\int_{\tilde{\Omega}} \nabla \tilde{u}_m w \,\mathrm{d}x = -\int_{\tilde{\Omega}} \tilde{u}_m \nabla w \,\mathrm{d}x = -\int_\Omega \tilde{u}_m \nabla w \,\mathrm{d}x. \tag{16}$$

Therefore the weak derivative of $\tilde{u}_m$ is in $L_2(\Omega)$, which implies that indeed $\tilde{u}_m \in H^1(\Omega, S^1)$.

The chain of inequalities in (13) now implies that

$$E^\infty(\tilde{u}_m) \leqslant E^\infty(u_m), \tag{17}$$

where both functions $\tilde{u}_m$ and u_m are in the space $H^1(\Omega, S^1)$. According to White's theory [8] the homotopy type is preserved through weak convergence, and thus $\tilde{u}_m$ is of m homotopy type. Therefore, the inequality (17) must be an equality. Hence $\tilde{u}_m = u_m$ (up to the usual irrelevant constant phase shift). This forces equalities everywhere in (13). Therefore the $L_2(\tilde{\Omega})$ norm of u_m^b converges to the $L_2(\tilde{\Omega})$ norm of $\tilde{u}_m$, and thus the sequence u_m^b converges strongly to u_m which completes the proof of the theorem. □

4. Related results and summary

Theorem 1 was formulated in a somewhat restricted way. In fact, similar results hold for a number of generalizations as follows.

- The parameter b appears in front of both the term $\int_{\tilde{\Omega}}(|u^2|-1)^2\,\mathrm{d}x$ and the term $\int_\Gamma [u]^2\,\mathrm{d}x$. It is easy to see that we could have used two different large parameters and pass to the limit with a double-indexed sequence.
- The Ginzburg Landau model of superconductivity for a superconducting sample with a toroidal shape that is interrupted by a weak link was introduced in section 1. Consider the following special case of it:

$$G_J^b(u,A) = \int_{\tilde{\Omega}} ((|\nabla - \mathrm{i}A)u|^2 + b(|u|^2-1)^2)\,\mathrm{d}x + b\int_\Gamma [u]^2\,\mathrm{d}x + \int_{R^n} |\nabla \times A|^2\,\mathrm{d}x. \quad (18)$$

  We seek minimizers (u, A) of G_J^b in the space $H^1(\tilde{\Omega}) \times H^1_{\mathrm{div}}$, where H^1_{div} is the Sobolev space of a divergence-free vector field that is obtained from the closure of the space of C_0^∞ vector fields under the norm $\int_{R^n} |\nabla v|^2\,\mathrm{d}x$. By exactly the same method of the previous section, we can show that for each integer m and for b sufficiently large there exists a local minimizer $\left(u_m^b, A_m\right)$ of F^b, such that u_m^b is close (in the H^1 sense) to a function in $H^1(\Omega, S^1)$ which is of the m homotopy type. The supercurrents associated with these metastable solutions are persistent currents.

To summarize, we established that the persistent currents in rings interrupted by Josephson junctions are related to the homotopy types in the ring. This gives the mathematical foundation for the experimental observation of Goldman *et al* [3].

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